

A NOTE ON LOCAL BEHAVIOR OF EIGENFUNCTIONS OF THE SCHRÖDINGER OPERATOR

IHYEOK SEO

ABSTRACT. We show that a real eigenfunction of the Schrödinger operator changes sign near some point in \mathbb{R}^n under a suitable assumption on the potential.

1. INTRODUCTION

The time evolution of a non-relativistic quantum particle is described by the wave function $\Psi(t, x)$ which is governed by the Schrödinger equation

$$i\partial_t \Psi(t, x) = H\Psi(t, x),$$

where the Hamiltonian $H = -\Delta + V(x)$ is called the Schrödinger operator. Here, Δ is the Laplace operator and V is a potential.

The fundamental approach to find a solution of the above equation is by separation of variables. In fact, considering the ansatz $\Psi(t, x) = f(t)\psi(x)$, the solution can be written as $\Psi(t, x) = f(0)e^{-iEt}\psi(x) = e^{-iEt}\Psi(0, x)$, where E is an eigenvalue with the corresponding eigenfunction ψ which is a solution of the following eigenvalue equation for the Schrödinger operator:

$$(-\Delta + V(x))\psi(x) = E\psi(x). \quad (1.1)$$

From the physical point of view, $E \in \mathbb{R}$ is the energy level of the particle.

In this note we are interested in local behavior of ψ near some point in \mathbb{R}^n . By using Brownian motion ideas, it was shown in [5] that for a certain class of potentials V , if $\psi(x_0) = 0$ for $x_0 \in \mathbb{R}^n$ and ψ is real, then either

- (a) ψ is identically zero near x_0 or
- (b) ψ has both positive and negative signs arbitrarily close to x_0 .

As remarked in [5], this asserts that the nodal set $\{x : \psi(x) = 0\}$ must have (Hausdorff) dimension at least $n - 1$. Also, in many cases, the first cannot occur if $\psi \not\equiv 0$, and so one can assert that the eigenfunction ψ changes sign near x_0 in that case.

Here we will consider potentials V given by

$$\|V\| := \sup_Q \left(\int_Q |V(x)| dx \right)^{-1} \int_Q \int_Q \frac{|V(x)V(y)|}{|x - y|^{n-2}} dx dy < \infty, \quad (1.2)$$

where the sup is taken over all dyadic cubes Q in \mathbb{R}^n , $n \geq 3$. Our goal is to prove the following theorem.

2010 *Mathematics Subject Classification.* Primary: 34L10; Secondary: 35J10.

Key words and phrases. Eigenfunctions, Schrödinger operator.

Theorem 1.1. *Let $n \geq 3$. Assume that $\psi \in H^1(\mathbb{R}^n)$ is real and is an eigenfunction of (1.1) with $E \in \mathbb{C}$. If the potential V satisfies (1.2) and ψ has a zero of infinite order at $x_0 \in \mathbb{R}^n$, then either (a) holds or (b) holds.*

Let us now give more details about the assumptions in the theorem.

First, $H^1(\mathbb{R}^n)$ denotes the Sobolev space of functions whose derivatives up to order 1 belong to $L^2(\mathbb{R}^n)$, and by an eigenfunction of (1.1) we mean a weak solution such that for every $\phi \in H_0^1(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \nabla \psi \cdot \nabla \phi + (V(x) - E)\psi\phi \, dx = 0. \quad (1.3)$$

Next, we say that ψ has a zero of infinite order at $x_0 \in \mathbb{R}^n$ if for all $m > 0$

$$\int_{B(x_0, \varepsilon)} \psi(x) \, dx = O(\varepsilon^m) \quad \text{as } \varepsilon \rightarrow 0, \quad (1.4)$$

where $B(x_0, \varepsilon)$ is the ball centered at x_0 with radius ε . For smooth ψ , (1.4) holds if and only if $D^\alpha \psi(x_0) = 0$ for every order $|\alpha|$.

Finally, the condition (1.2) is closely related to the global Kato and Rollnik potential classes, denoted by \mathcal{K} and \mathcal{R} , respectively, which are defined by

$$V \in \mathcal{K} \quad \Leftrightarrow \quad \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|V(y)|}{|x - y|^{n-2}} \, dy < \infty$$

and

$$V \in \mathcal{R} \quad \Leftrightarrow \quad \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(x)V(y)|}{|x - y|^2} \, dx \, dy < \infty.$$

These are fundamental classes of potentials in spectral and scattering theory. Indeed, it is not difficult to see that the Kato and Rollnik potentials satisfy the condition (1.2). It should be also noted that there are potentials satisfying (1.2) which are not in \mathcal{K} . For example, $V(x) = 1/|x|^2$. More generally, potentials in the Fefferman-Phong class \mathcal{F}^p , which is defined by

$$V \in \mathcal{F}^p \quad \Leftrightarrow \quad \sup_{x, r} r^{2-n/p} \left(\int_{B(x, r)} |V(y)|^p \, dy \right)^{1/p} < \infty$$

for $1 < p \leq n/2$, satisfy (1.2) (see, for example, [8]). Note that $L^{n/2} = \mathcal{F}^{n/2}$ and $1/|x|^2 \in L^{n/2, \infty} \subset \mathcal{F}^p$ if $p \neq n/2$. We also point out that the above theorem can be found in [2] for $V \in L^{n/2, \infty}$.

2. PRELIMINARIES

The key ingredient in the proof of the theorem is the following lemma, due to Kerman and Sawyer [7] (see Theorem 2.3 there and also Lemma 2.1 in [1]), which characterizes weighted L^2 inequalities for the fractional integral

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} \, dy, \quad 0 < \alpha < n.$$

Lemma 2.1. *Let $n \geq 3$. Assume that w be a nonnegative measurable function on \mathbb{R}^n . Then there exists a constant C_w depending on w such that the following inequality*

$$\|I_{\alpha/2}f\|_{L^2(w)} \leq C_w \|f\|_{L^2} \quad (2.1)$$

holds for all measurable functions f on \mathbb{R}^n if and only if

$$\|w\|_\alpha := \sup_Q \left(\int_Q w(x) dx \right)^{-1} \int_Q \int_Q \frac{w(x)w(y)}{|x-y|^{n-\alpha}} dx dy < \infty. \quad (2.2)$$

Furthermore, the constant C_w may be taken to be a constant multiple of $\|w\|_\alpha^{1/2}$.

To prove Theorem 1.1 in the next section, we will use the above lemma with $\alpha = 2$. (Recall that the condition (1.2) corresponds to the case $\alpha = 2$ in (2.2).) Also, the following simple lemma is needed for handling the energy constant E .

Lemma 2.2. *Let $\chi_{B(x_0, r)}$ be the characteristic function of the ball $B(x_0, r) \subset \mathbb{R}^n$. Then there exists $r_0 > 0$ such that $w = \chi_{B(x_0, r)}$ satisfies (2.2) uniformly for all $r < r_0$.*

Proof. First, note that

$$\sup_Q \left(\int_Q w(x) dx \right)^{-1} \int_Q \int_Q \frac{w(x)w(y)}{|x-y|^{n-\alpha}} dx dy \leq \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{w(y)}{|x-y|^{n-\alpha}} dy.$$

Hence, it suffices to show that

$$\lim_{r \rightarrow 0} \sup_{x \in \mathbb{R}^n} \int_{|y-x_0| < r} \frac{1}{|x-y|^{n-\alpha}} dy = 0.$$

But, this is an easy consequence of the following computation:

$$\begin{aligned} \sup_{x \in \mathbb{R}^n} \int_{|y-x_0| < r} \frac{1}{|x-y|^{n-\alpha}} dy &\leq \sup_{|x-x_0| < 2r} \int_{|y-x_0| < r} |x-y|^{-(n-\alpha)} dy \\ &\quad + \sup_{|x-x_0| \geq 2r} \int_{|y-x_0| < r} r^{-(n-\alpha)} dy \\ &\leq \sup_{x \in \mathbb{R}^n} \int_{|x-y| < 4r} |x-y|^{-(n-\alpha)} dy + Cr^\alpha \\ &\leq Cr^\alpha. \end{aligned}$$

□

Finally, we recall the following lemma (see, for example, [4]) concerning the doubling property.

Lemma 2.3. *Let $f \in L^1_{loc}$ be a function in a ball $B(x_0, r_0) \subset \mathbb{R}^n$. Assume that the doubling property*

$$\int_{B(x_0, 2r)} f dx \leq C \int_{B(x_0, r)} f dx \quad (2.3)$$

holds for all r with $2r < r_0$. If $f \geq 0$ and f has a zero of infinite order at x_0 , then f must vanish identically in $B(x_0, r_0)$.

3. PROOF OF THEOREM 1.1

Suppose that (b) does not hold. Without loss of generality, we may then assume $\psi \geq 0$ near x_0 (since the other case $\psi \leq 0$ follows clearly from the same argument). With this assumption, we will show that ψ must vanish identically in a sufficiently small neighborhood of x_0 , and thereby we prove the theorem. For simplicity of notation we shall also assume $x_0 = 0$, and we will use the letter C to denote positive constants possibly different at each occurrence.

Now, let η be a smooth function supported in $B(0, 2\delta)$ such that $0 \leq \eta \leq 1$, $\eta = 1$ on $B(0, \delta)$ and $|\nabla \eta| \leq C\delta^{-1}$. Here, $\delta > 0$ is less than a fixed δ_0 chosen later. Putting $\phi = \eta^2/(\psi + \varepsilon)$ with $\varepsilon > 0$ in the integral in (1.3), we see that

$$\int \frac{2\eta}{\psi + \varepsilon} \nabla \psi \cdot \nabla \eta \, dx - \int \frac{\eta^2}{(\psi + \varepsilon)^2} \nabla \psi \cdot \nabla \psi \, dx + \int (V(x) - E) \frac{\psi \eta^2}{\psi + \varepsilon} \, dx = 0.$$

(Here it is an elementary matter to check $\phi \in H_0^1$.) By setting $\tilde{\psi} = \ln(\psi + \varepsilon)$, it follows now that

$$\int 2\eta \nabla \tilde{\psi} \cdot \nabla \eta \, dx - \int \eta^2 \nabla \tilde{\psi} \cdot \nabla \tilde{\psi} \, dx + \int (V(x) - E) \frac{\psi \eta^2}{\psi + \varepsilon} \, dx = 0. \quad (3.1)$$

Using the simple algebraic inequality

$$2ab \leq (a^2/4 + 4b^2), \quad a, b \geq 0,$$

we bound the first integral in (3.1) as follows:

$$\begin{aligned} \left| \int 2\eta \nabla \tilde{\psi} \cdot \nabla \eta \, dx \right| &\leq \int 2|\eta \nabla \tilde{\psi}| |\nabla \eta| \, dx \\ &\leq \int \frac{1}{4} \eta^2 |\nabla \tilde{\psi}|^2 \, dx + \int 4|\nabla \eta|^2 \, dx. \end{aligned}$$

Then by combining this and (3.1), it is not difficult to see that

$$\int \eta^2 |\nabla \tilde{\psi}|^2 \, dx \leq \frac{16}{3} \int |\nabla \eta|^2 \, dx + \frac{4}{3} \int |V(x) - E| \eta^2 \, dx. \quad (3.2)$$

Now, using Lemmas 2.1 and 2.2 in the previous section, the second term in the right-hand side of (3.2) is bounded as follows:

$$\begin{aligned} \int |V(x) - E| \eta^2 \, dx &\leq \int |V| \eta^2 \, dx + |E| \int \chi_{B(0, 2\delta)} \eta^2 \, dx \\ &\leq C\|V\| \int |\nabla \eta|^2 \, dx + C|E| \|\chi_{B(0, 2\delta)}\| \int |\nabla \eta|^2 \, dx \\ &\leq C(\|V\| + 1) \int |\nabla \eta|^2 \, dx \end{aligned}$$

if $2\delta < \delta_0$ for a sufficiently small δ_0 . Indeed, note that when $\alpha = 2$ in Lemma 2.1, the inequality (2.1) is equivalent to

$$\int |g|^2 w \, dx \leq C\|w\| \int |\nabla g|^2 \, dx, \quad g \in H^1.$$

Then this and Lemma 2.2 give the above bound. Consequently, returning to (3.2) and recalling $\eta = 1$ on $B(0, \delta)$, we get

$$\begin{aligned} \int_{B(0, \delta)} |\nabla \tilde{\psi}|^2 dx &\leq \int \eta^2 |\nabla \tilde{\psi}|^2 dx \leq C \int |\nabla \eta|^2 dx \\ &\leq C \int_{B(0, 2\delta)} \delta^{-2} dx \\ &\leq C \delta^{n-2}. \end{aligned} \quad (3.3)$$

At this point, one can apply the Poincaré inequality and the lemma of John and Nirenberg [6], as in [2], in order to conclude that for a small $\rho > 0$

$$\left(\frac{1}{|B(0, \delta)|} \int_{B(0, \delta)} e^{\rho \tilde{\psi}} dx \right) \left(\frac{1}{|B(0, \delta)|} \int_{B(0, \delta)} e^{-\rho \tilde{\psi}} dx \right) < C. \quad (3.4)$$

In fact, by the Poincaré inequality and (3.3),

$$\int_{B(0, \delta)} |\tilde{\psi} - \tilde{\psi}_B|^2 dx \leq C \delta^2 \int_{B(0, \delta)} |\nabla \tilde{\psi}|^2 dx \leq C \delta^n,$$

where

$$\tilde{\psi}_B = \frac{1}{|B(0, \delta)|} \int_{B(0, \delta)} \tilde{\psi} dx.$$

Now, by Hölder's inequality

$$\int_{B(0, \delta)} |\tilde{\psi} - \tilde{\psi}_B| dx \leq C \delta^n,$$

and so $\tilde{\psi}$ belongs to the BMO space (in $B(0, \delta_0)$). Thus, by the lemma of John and Nirenberg [6], there exists a small $\rho > 0$ so that (3.4) holds. Since $\tilde{\psi} = \ln(\psi + \varepsilon)$, by Fatou's lemma, (3.4) leads to

$$\left(\frac{1}{|B(0, \delta)|} \int_{B(0, \delta)} \psi^\rho dx \right) \left(\frac{1}{|B(0, \delta)|} \int_{B(0, \delta)} \psi^{-\rho} dx \right) < C.$$

It is a well known fact¹ (see [3]) that this implies the doubling property (2.3) for ψ^ρ . Then, by Lemma 2.3, ψ^ρ must vanish identically near $x_0 = 0$, and so $\psi \equiv 0$ near $x_0 = 0$.

REFERENCES

- [1] J. A. Barcelo, J. M. Bennett, A. Ruiz and M. C. Vilela, *Local smoothing for Kato potentials in three dimensions*, Math. Nachr. 282 (2009), 1391-1405.
- [2] F. Chiarenza and N. Garofalo, *Unique continuation for nonnegative solutions of Schrödinger operators*, Preprint Series No. 122, University of Minnesota, 1984.
- [3] R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. 51 (1974), 241-250.
- [4] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Annals of Math. Studies, no. 105, Princeton Univ. Press (1983).

¹The doubling property is satisfied for functions in the A_2 Muckenhoupt class.

- [5] M. Hoffman-Ostenhof, T. Hoffman-Ostenhof and B. Simon, *Brownian motion and a consequence of Harnack's inequality: nodes of quantum wave functions*, Proc. Amer. Math. Soc. 80 (1980), 301-305.
- [6] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. 14 (1961), 415-426.
- [7] R. Kerman and E. Sawyer, *The trace inequality and eigenvalue estimates for Schrödinger operators*, Ann. Inst. Fourier (Grenoble) 36 (1986), 207-228.
- [8] I. Seo, *On minimal support properties of solutions of Schrödinger equations*, J. Math. Anal. Appl. 414 (2014), 21-28.

SCHOOL OF MATHEMATICS, KOREA INSTITUTE FOR ADVANCED STUDY, SEOUL 130-722, REPUBLIC OF KOREA

E-mail address: `ihseo@kias.re.kr`